Volumes by Cylindrical Shells

Some volume problems are very difficult to handle by the methods of Section 6.2. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.) If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

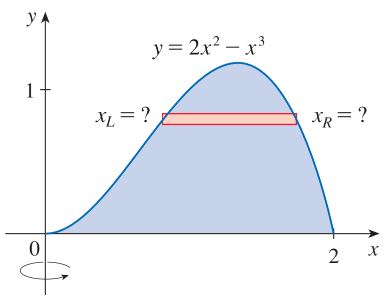


FIGURE 1

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

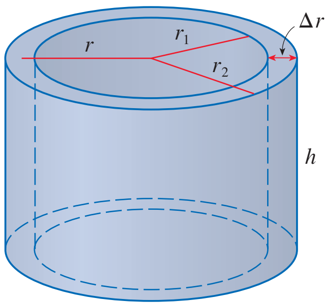


FIGURE 2

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \end{aligned}$$

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

1 $$\boxed{V = 2\pi r h\, \Delta r}$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geqslant 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geqslant 0$. (See Figure 3.)

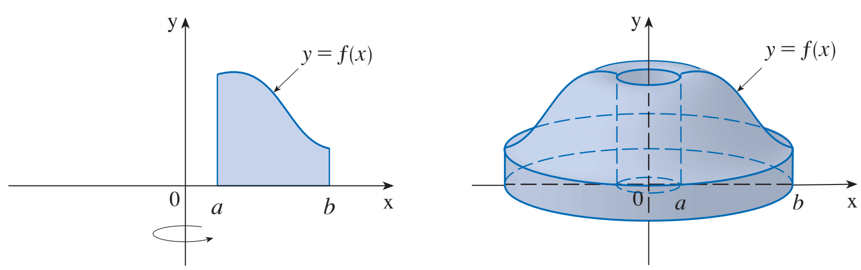


FIGURE 3

We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx (see Figure 4), so by Formula 1 its volume is

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

Therefore, an approximation to the volume V of S is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x$$

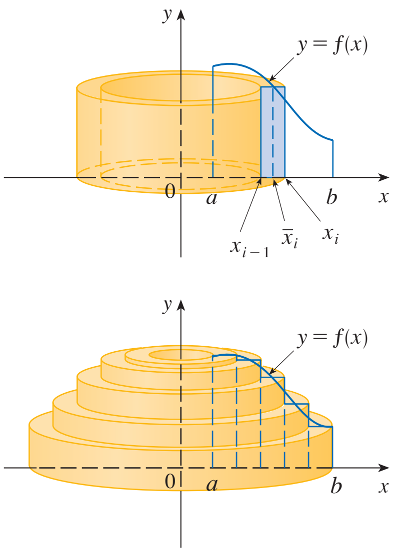


FIGURE 4

This approximation appears to become better as $n \to \infty$. But, from the definition of an integral, we know that

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\, \Delta x = \int_a^b 2\pi x f(x)\, dx$$

Thus, the following appears plausible:

> $\boxed{2}$ The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is
>
> $$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

The argument using cylindrical shells makes Formula 2 seem reasonable, but later we will be able to prove it. (See Exercise 47.)

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \; \underbrace{[f(x)]}_{\text{height}} \, dx$$

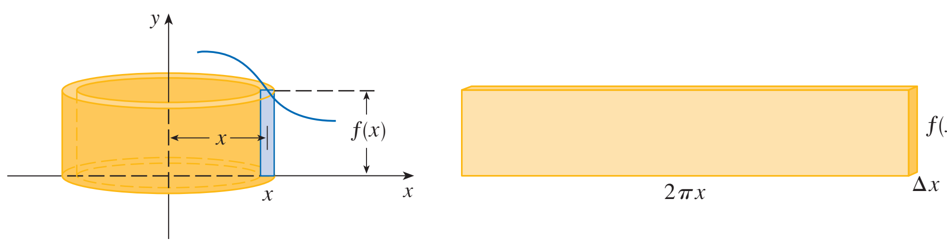


FIGURE 5

This type of reasoning will be helpful in other situations, such as when we rotate about lines other than the y-axis.

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

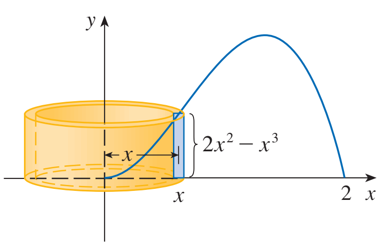


FIGURE 6

SOLUTION From the sketch in Figure 6 we see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$. So, by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\, dx = 2\pi \int_0^2 (2x^3 - x^4)\, dx$$

$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2 = 2\pi\left(8 - \tfrac{32}{5}\right) = \tfrac{16}{5}\pi$$

It can be verified that the shell method gives the same answer as slicing. ■■

■■ Figure 7 shows a computer-generated picture of the solid whose volume we computed in Example 1.

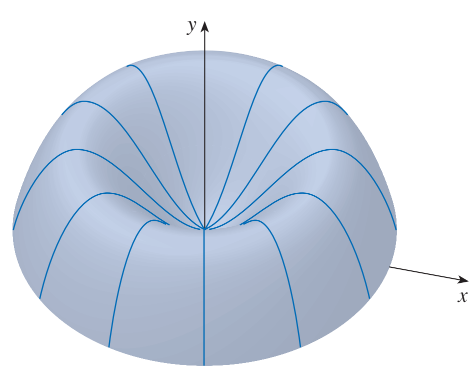


FIGURE 7

NOTE ▫ Comparing the solution of Example 1 with the remarks at the beginning of this section, we see that the method of cylindrical shells is much easier than the washer method for this problem. We did not have to find the coordinates of the local maximum and we did not have to solve the equation of the curve for x in terms of y. However, in other examples the methods of the preceding section may be easier.

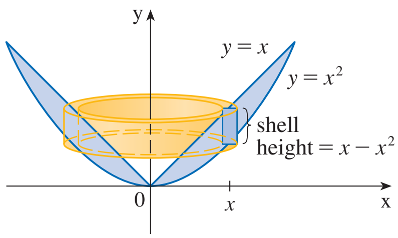


FIGURE 8

EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

SOLUTION The region and a typical shell are shown in Figure 8. We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$
$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

As the following example shows, the shell method works just as well if we rotate about the x-axis. We simply have to draw a diagram to identify the radius and height of a shell.

EXAMPLE 3 Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

SOLUTION This problem was solved using disks in Example 2 in Section 6.2. To use shells we relabel the curve $y = \sqrt{x}$ (in the figure in that example) as $x = y^2$ in Figure 9. For rotation about the x-axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

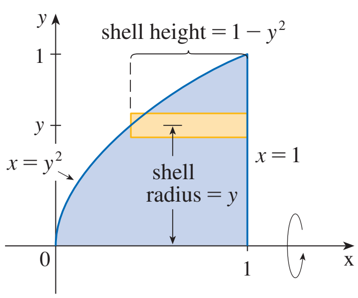


FIGURE 9

$$V = \int_0^1 (2\pi y)(1 - y^2)\,dy = 2\pi \int_0^1 (y - y^3)\,dy$$
$$= 2\pi\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

In this problem the disk method was simpler.

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

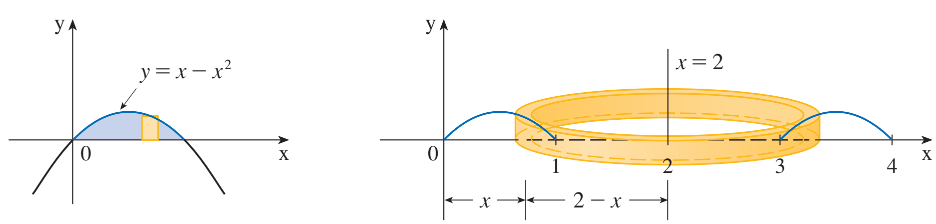


FIGURE 10

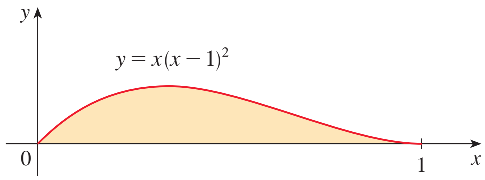
The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

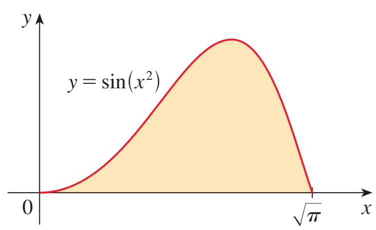
Exercises

A Click here for answers.   S Click here for solutions.

1. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use slicing to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use shells to find V.



2. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Sketch a typical cylindrical shell and find its circumference and height. Use shells to find the volume of S. Do you think this method is preferable to slicing? Explain.



3–7 ■ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

3. $y = 1/x, \quad y = 0, \quad x = 1, \quad x = 2$

4. $y = x^2, \quad y = 0, \quad x = 1$

5. $y = e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1$

6. $y = 3 + 2x - x^2, \quad x + y = 3$

7. $y = 4(x-2)^2, \quad y = x^2 - 4x + 7$

8. Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sqrt{x}$ and $y = x^2$. Find V both by slicing and by cylindrical shells. In both cases draw a diagram to explain your method.

9–14 ■ Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis. Sketch the region and a typical shell.

9. $x = 1 + y^2, \quad x = 0, \quad y = 1, \quad y = 2$

10. $x = \sqrt{y}, \quad x = 0, \quad y = 1$

11. $y = x^3, \quad y = 8, \quad x = 0$

12. $x = 4y^2 - y^3, \quad x = 0$

13. $y = 4x^2, \quad 2x + y = 6$

14. $x + y = 3, \quad x = 4 - (y-1)^2$

15–20 ■ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

15. $y = x^2, \ y = 0, \ x = 1, \ x = 2; \quad$ about $x = 1$

16. $y = x^2, \ y = 0, \ x = -2, \ x = -1; \quad$ about the y-axis

17. $y = x^2, \ y = 0, \ x = 1, \ x = 2; \quad$ about $x = 4$

18. $y = 4x - x^2, \ y = 8x - 2x^2; \quad$ about $x = -2$

19. $y = \sqrt{x-1}, \ y = 0, \ x = 5; \quad$ about $y = 3$

20. $y = x^2, \ x = y^2; \quad$ about $y = -1$

21–26 ■ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

21. $y = \ln x, \ y = 0, \ x = 2; \quad$ about the y-axis

22. $y = x, \ y = 4x - x^2; \quad$ about $x = 7$

23. $y = x^4, \ y = \sin(\pi x/2); \quad$ about $x = -1$

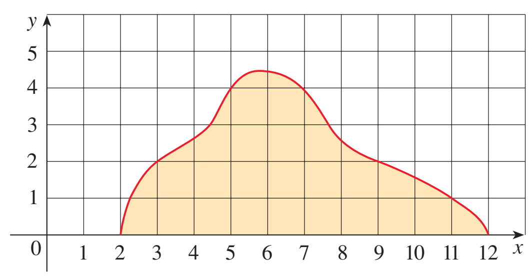
24. $y = 1/(1 + x^2), \ y = 0, \ x = 0, \ x = 2; \quad$ about $x = 2$

25. $x = \sqrt{\sin y}, \ 0 \leqslant y \leqslant \pi, \ x = 0; \quad$ about $y = 4$

26. $x^2 - y^2 = 7, \ x = 4; \quad$ about $y = 5$

27. Use the Midpoint Rule with $n = 4$ to estimate the volume obtained by rotating about the y-axis the region under the curve $y = \tan x$, $0 \leqslant x \leqslant \pi/4$.

28. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 5$ to estimate the volume of the solid.



29–32 ■ Each integral represents the volume of a solid. Describe the solid.

29. $\displaystyle\int_0^3 2\pi x^5 \, dx$

30. $\displaystyle 2\pi \int_0^2 \frac{y}{1 + y^2} \, dy$

31. $\displaystyle\int_0^1 2\pi(3 - y)(1 - y^2) \, dy$

32. $\displaystyle\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x) \, dx$

33–34 ■ Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

33. $y = 0, \quad y = x + x^2 - x^4$

34. $y = x^4, \quad y = 3x - x^3$

CAS **35–36** ■ Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

35. $y = \sin^2 x,\ y = \sin^4 x,\ 0 \leqslant x \leqslant \pi;$ about $x = \pi/2$

36. $y = x^3 \sin x,\ y = 0,\ 0 \leqslant x \leqslant \pi;$ about $x = -1$

37–42 ■ The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

37. $y = x^2 + x - 2,\ y = 0;$ about the x-axis

38. $y = x^2 - 3x + 2,\ y = 0;$ about the y-axis

39. $y = 5,\ y = x + (4/x);$ about $x = -1$

40. $x = 1 - y^4,\ x = 0;$ about $x = 2$

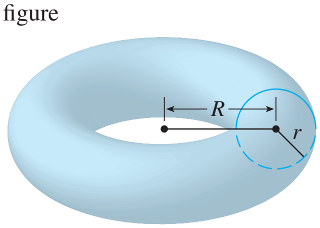
41. $x^2 + (y - 1)^2 = 1;$ about the y-axis

42. $x^2 + (y - 1)^2 = 1;$ about the x-axis

43–45 ■ Use cylindrical shells to find the volume of the solid.

43. A sphere of radius r

44. The solid torus (a donut-shaped solid with radii R and r) shown in the figure

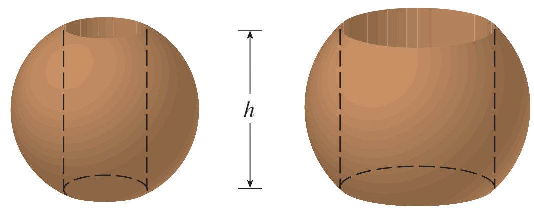


45. A right circular cone with height h and base radius r

46. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.

(a) Guess which ring has more wood in it.

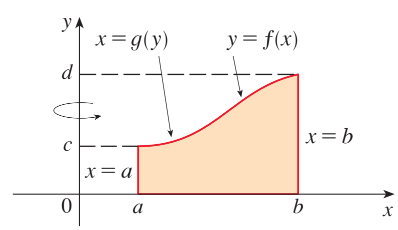
(b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.



47. We arrived at Formula 2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2\,dy$$

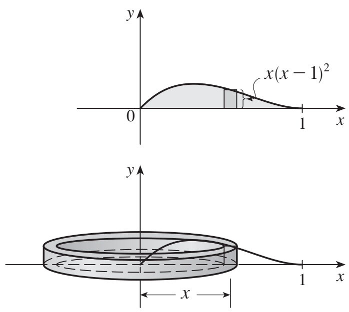
Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that $V = \int_a^b 2\pi x f(x)\,dx$.



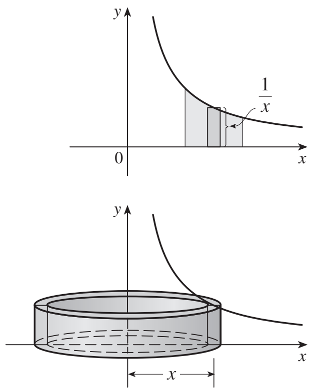
Answers

S Click here for solutions.

1. Circumference $= 2\pi x$, height $= x(x-1)^2$; $\pi/15$

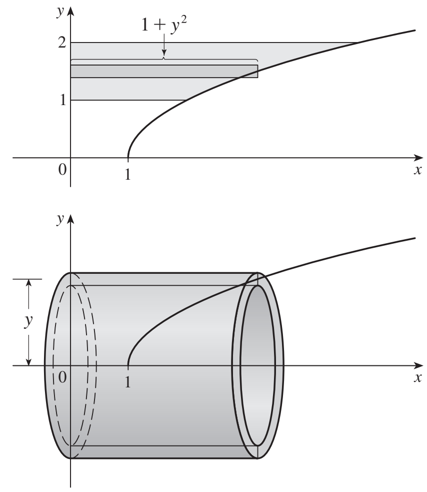


3. 2π



5. $\pi(1 - 1/e)$ **7.** 16π

9. $21\pi/2$



11. $768\pi/7$ **13.** $250\pi/3$

15. $17\pi/6$ **17.** $67\pi/6$ **19.** 24π

21. $\int_1^2 2\pi x \ln x\, dx$

23. $\int_0^1 2\pi(x+1)[\sin(\pi x/2) - x^4]\, dx$

25. $\int_0^\pi 2\pi(4-y)\sqrt{\sin y}\, dy$ **27.** 1.142

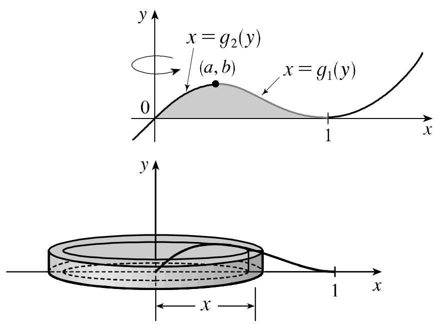
29. Solid obtained by rotating the region $0 \leqslant y \leqslant x^4$, $0 \leqslant x \leqslant 3$ about the y-axis

31. Solid obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$, or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$

33. 0, 1.32; 4.05 **35.** $\frac{1}{32}\pi^3$ **37.** $81\pi/10$

39. $8\pi(3 - \ln 4)$ **41.** $4\pi/3$ **43.** $\frac{4}{3}\pi r^3$ **45.** $\frac{1}{3}\pi r^2 h$

Solutions: Volumes by Cylindrical Shells

1. 

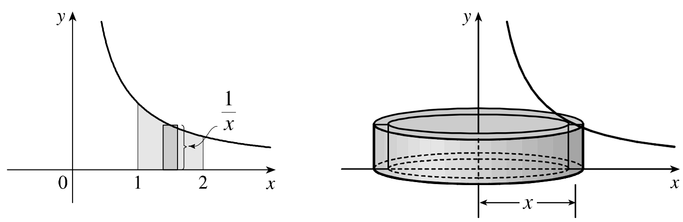
If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x\left[x(x-1)^2\right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 1\, dx$

$= 2\pi \left[x\right]_1^2 = 2\pi(2-1) = 2\pi$

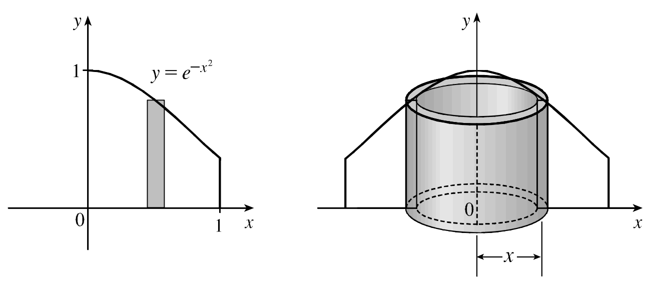


5. $V = \int_0^1 2\pi x e^{-x^2}\, dx$. Let $u = x^2$.

Thus, $du = 2x\, dx$, so

$V = \pi \int_0^1 e^{-u}\, du = \pi\left[-e^{-u}\right]_0^1$

$= \pi(1 - 1/e)$



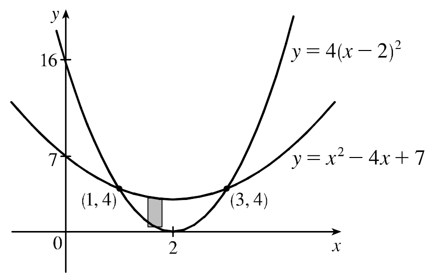
7. The curves intersect when $4(x-2)^2 = x^2 - 4x + 7 \quad\Leftrightarrow\quad 4x^2 - 16x + 16 = x^2 - 4x + 7 \quad\Leftrightarrow$

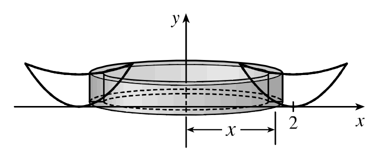
$3x^2 - 12x + 9 = 0 \quad\Leftrightarrow\quad 3(x^2 - 4x + 3) = 0 \quad\Leftrightarrow\quad 3(x-1)(x-3) = 0$, so $x = 1$ or 3.

$$V = 2\pi \int_1^3 \left\{x\left[\left(x^2 - 4x + 7\right) - 4(x-2)^2\right]\right\} dx = 2\pi \int_1^3 \left[x(x^2 - 4x + 7 - 4x^2 + 16x - 16)\right] dx$$

$$= 2\pi \int_1^3 \left[x(-3x^2 + 12x - 9)\right] dx = 2\pi(-3)\int_1^3 (x^3 - 4x^2 + 3x)\, dx = -6\pi\left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_1^3$$

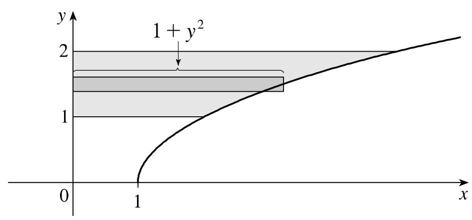
$$= -6\pi\left[\left(\tfrac{81}{4} - 36 + \tfrac{27}{2}\right) - \left(\tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \tfrac{4}{3}\right) = -6\pi\left(-\tfrac{8}{3}\right) = 16\pi$$

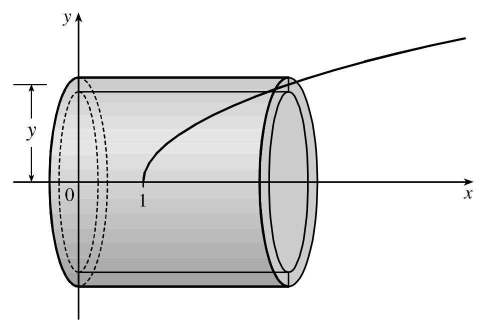




9. $V = \int_1^2 2\pi y\left(1 + y^2\right) dy = 2\pi \int_1^2 \left(y + y^3\right) dy = 2\pi\left[\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4\right]_1^2$

$= 2\pi\left[(2 + 4) - \left(\tfrac{1}{2} + \tfrac{1}{4}\right)\right] = 2\pi\left(\tfrac{21}{4}\right) = \tfrac{21\pi}{2}$

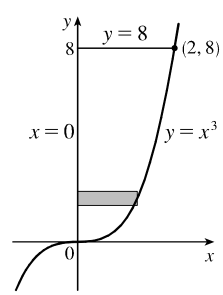


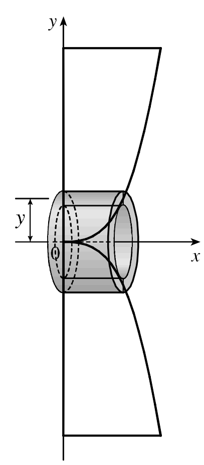


11. $V = 2\pi \displaystyle\int_0^8 \left[y(\sqrt[3]{y} - 0)\right] dy$

$$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi\left[\tfrac{3}{7}y^{7/3}\right]_0^8$$

$$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$$





13. The curves intersect when $4x^2 = 6 - 2x \quad \Leftrightarrow \quad 2x^2 + x - 3 = 0 \quad \Leftrightarrow \quad (2x+3)(x-1) = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2}$ or 1.

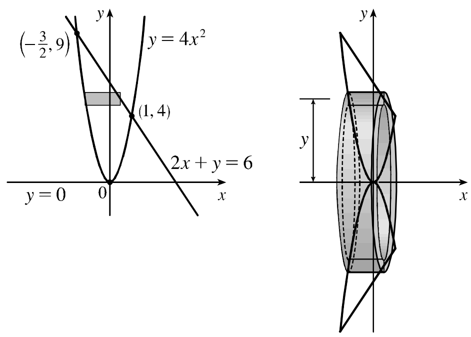
Solving the equations for x gives us $y = 4x^2 \quad \Rightarrow \quad x = \pm\frac{1}{2}\sqrt{y}$ and $2x + y = 6 \quad \Rightarrow \quad x = -\frac{1}{2}y + 3$.

$$V = 2\pi \int_0^4 \left\{y\left[\left(\tfrac{1}{2}\sqrt{y}\right) - \left(-\tfrac{1}{2}\sqrt{y}\right)\right]\right\} dy + 2\pi \int_4^9 \left\{y\left[\left(-\tfrac{1}{2}y + 3\right) - \left(-\tfrac{1}{2}\sqrt{y}\right)\right]\right\} dy$$

$$= 2\pi \int_0^4 \left(y\sqrt{y}\right) dy + 2\pi \int_4^9 \left(-\tfrac{1}{2}y^2 + 3y + \tfrac{1}{2}y^{3/2}\right) dy = 2\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^4 + 2\pi\left[-\tfrac{1}{6}y^3 + \tfrac{3}{2}y^2 + \tfrac{1}{5}y^{5/2}\right]_4^9$$

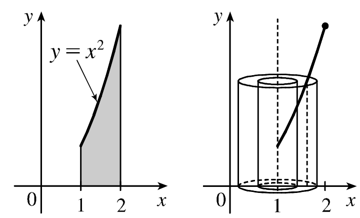
$$= 2\pi\left(\tfrac{2}{5} \cdot 32\right) + 2\pi\left[\left(-\tfrac{243}{2} + \tfrac{243}{2} + \tfrac{243}{5}\right) - \left(-\tfrac{32}{3} + 24 + \tfrac{32}{5}\right)\right]$$

$$= \tfrac{128}{5}\pi + 2\pi\left(\tfrac{433}{15}\right) = \tfrac{1250}{15}\pi = \tfrac{250}{3}\pi$$



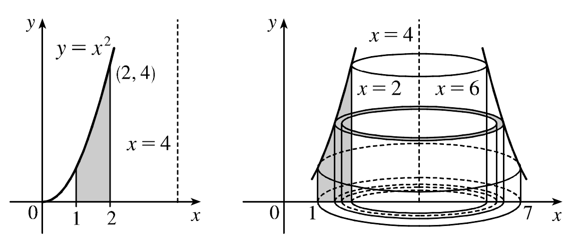
15. $V = \int_1^2 2\pi(x-1)x^2\,dx = 2\pi\left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_1^2$

$= 2\pi\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = \frac{17}{6}\pi$



17. $V = \int_1^2 2\pi(4-x)x^2\,dx = 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^2$

$= 2\pi\left[\left(\frac{32}{3} - 4\right) - \left(\frac{4}{3} - \frac{1}{4}\right)\right] = \frac{67}{6}\pi$



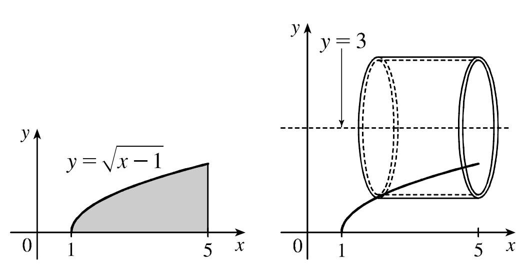
19. $V = \int_0^2 2\pi(3-y)(5-x)dy$

$= \int_0^2 2\pi(3-y)\left(5 - y^2 - 1\right) dy$

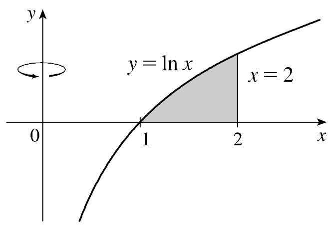
$= \int_0^2 2\pi\left(12 - 4y - 3y^2 + y^3\right) dy$

$= 2\pi\left[12y - 2y^2 - y^3 + \frac{1}{4}y^4\right]_0^2$

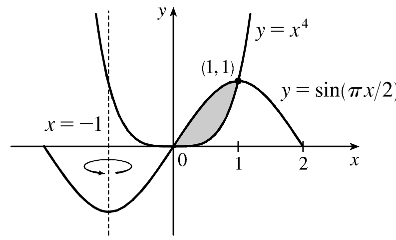
$= 2\pi(24 - 8 - 8 + 4) = 24\pi$



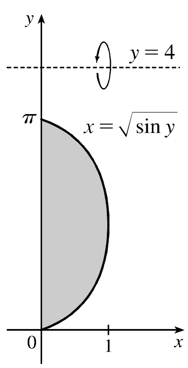
21. $V = \int_1^2 2\pi x \ln x\, dx$



23. $V = \int_0^1 2\pi[x - (-1)]\left(\sin \frac{\pi}{2}x - x^4\right) dx$



25. $V = \int_0^\pi 2\pi(4 - y)\sqrt{\sin y}\, dy$



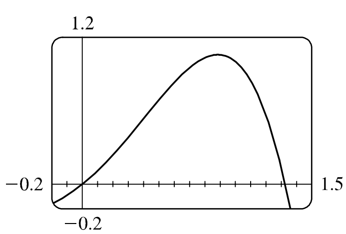
27. $\Delta x = \dfrac{\pi/4 - 0}{4} = \dfrac{\pi}{16}$.

$V = \int_0^{\pi/4} 2\pi x \tan x\, dx \approx 2\pi \cdot \frac{\pi}{16}\left(\frac{\pi}{32}\tan\frac{\pi}{32} + \frac{3\pi}{32}\tan\frac{3\pi}{32} + \frac{5\pi}{32}\tan\frac{5\pi}{32} + \frac{7\pi}{32}\tan\frac{7\pi}{32}\right) \approx 1.142$

29. $\int_0^3 2\pi x^5\, dx = 2\pi \int_0^3 x(x^4)\, dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

31. $\int_0^1 2\pi(3 - y)(1 - y^2)\, dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

33.

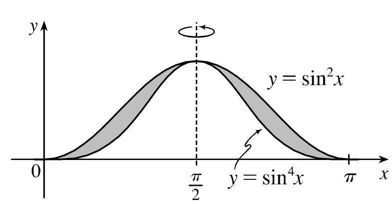


From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.32$, with $x + x^2 - x^4 > 0$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a \left[x(x + x^2 - x^4)\right] dx = 2\pi \int_0^a (x^2 + x^3 - x^5)\, dx$$
$$= 2\pi\left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^a \approx 4.05$$

35. $V = 2\pi \displaystyle\int_0^{\pi/2} \left[\left(\tfrac{\pi}{2} - x\right)\left(\sin^2 x - \sin^4 x\right)\right] dx$

$\overset{\text{CAS}}{=} \frac{1}{32}\pi^3$



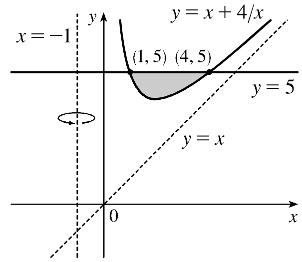
37. Use disks:

$$V = \int_{-2}^{1} \pi\left(x^2 + x - 2\right)^2 \, dx = \pi \int_{-2}^{1} \left(x^4 + 2x^3 - 3x^2 - 4x + 4\right) dx$$

$$= \pi\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - x^3 - 2x^2 + 4x\right]_{-2}^{1} = \pi\left[\left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 4\right) - \left(-\tfrac{32}{5} + 8 + 8 - 8 - 8\right)\right]$$

$$= \pi\left(\tfrac{33}{5} + \tfrac{3}{2}\right) = \tfrac{81}{10}\pi$$

39. Use shells:



$$V = \int_1^4 2\pi[x - (-1)][5 - (x + 4/x)] \, dx$$

$$= 2\pi \int_1^4 (x + 1)(5 - x - 4/x) \, dx$$

$$= 2\pi \int_1^4 \left(5x - x^2 - 4 + 5 - x - 4/x\right) dx$$

$$= 2\pi \int_1^4 \left(-x^2 + 4x + 1 - 4/x\right) dx = 2\pi\left[-\tfrac{1}{3}x^3 + 2x^2 + x - 4\ln x\right]_1^4$$

$$= 2\pi\left[\left(-\tfrac{64}{3} + 32 + 4 - 4\ln 4\right) - \left(-\tfrac{1}{3} + 2 + 1 - 0\right)\right]$$

$$= 2\pi(12 - 4\ln 4) = 8\pi(3 - \ln 4)$$

41. Use disks: $V = \pi \displaystyle\int_0^2 \left[\sqrt{1 - (y-1)^2}\right]^2 dy = \pi \int_0^2 \left(2y - y^2\right) dy = \pi\left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \pi\left(4 - \frac{8}{3}\right) = \frac{4}{3}\pi$

43. $V = 2\int_0^r 2\pi x\sqrt{r^2 - x^2} \, dx = -2\pi \int_0^r \left(r^2 - x^2\right)^{1/2}(-2x) \, dx = \left[-2\pi \cdot \tfrac{2}{3}\left(r^2 - x^2\right)^{3/2}\right]_0^r$

$$= -\tfrac{4}{3}\pi\left(0 - r^3\right) = \tfrac{4}{3}\pi r^3$$

45. $V = 2\pi \displaystyle\int_0^r x\left(-\frac{h}{r}x + h\right) dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right) dx = 2\pi h\left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r = 2\pi h\,\frac{r^2}{6} = \frac{\pi r^2 h}{3}$

47. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2 \, dy - \int_0^c \pi a^2 \, dy - \int_c^d \pi[g(y)]^2 \, dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2 \, dy$. Let $y = f(x)$, which gives $dy = f'(x) \, dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x) \, dx$. Now integrate by parts with $u = x^2$, and $dv = f'(x) \, dx \quad \Rightarrow \quad du = 2x \, dx$, $v = f(x)$, and

$\int_a^b x^2 f'(x) \, dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x \, f(x) \, dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x \, f(x) \, dx$, but $f(a) = c$ and $f(b) = d$

$\Rightarrow \quad V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x f(x) \, dx\right] = \int_a^b 2\pi x f(x) \, dx.$